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Relative projection frequency tables for orthogonal arrays

Relative Projektionshäufigkeitstabellen für orthogonale Felder (englischsprachig)

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Relative projection frequency tables for orthogonal arrays

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Abstract

Projection frequency tables provide detailed information on the confounding structure of an orthogonal array: they tabulate the frequency distribution of the numbers of (generalized) words of length 3 for all projections onto three factors (or length 4 for 4-factor projections or …). This article introduces relative projection frequency tables, which are more suitable than their absolute counterparts for assessing the severity of confounding in mixed level orthogonal arrays. Together with two scalar criteria derived from them, relative projection frequency tables can be used for implementing the newly-proposed relative projection aberration criterion, which ranks mixed level designs w.r.t. suitability for screening experiments.

1. Introduction

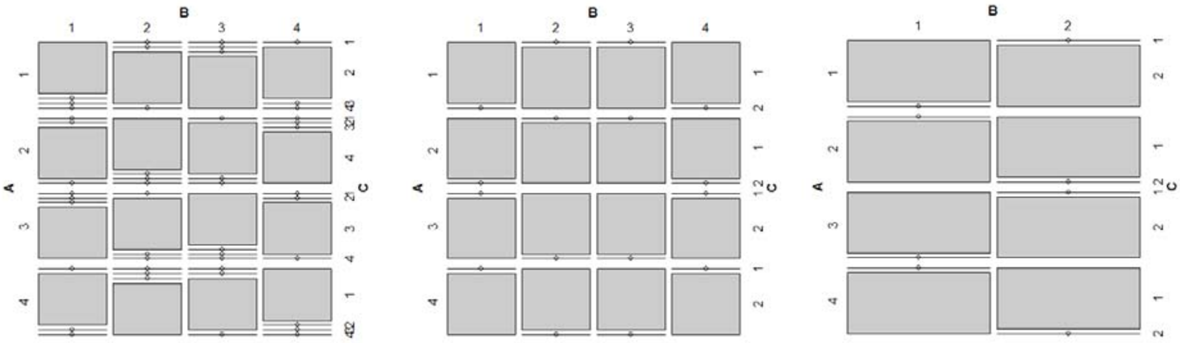
This article is devoted to general orthogonal arrays (OAs), including the mixed level situation. Considerations will be limited to arrays for qualitative factors, i.e. the factor levels are considered as unordered. Xu, Phoa and Wong (2009) gave an excellent overview over the state of the art regarding general OAs. The main purpose of the present article is to provide instruments – relative projection frequency tables and relative projection aberration – that are usable for assessing suitability of mixed level OAs for screening experiments. It will be demonstrated that the relative considerations add an important aspect over and above the known concepts of (absolute) projection frequency tables (Xu, Cheng and Wu 2004; cf. also next section).

An nxk array in n runs (=rows) and k factors (=columns) is an OA, if for each pair of columns each combination of entries occurs equally often. This imposes constraints on the possible combinations of n and k, depending on the numbers of levels of each of the k factors. In line with Hedayat, Sloane and Stufken (1999), OAs with all factors at the same number of levels will be called fixed level OAs, while OAs with factors at different numbers of levels will be called mixed level OAs.

Fixed level OAs with 2 levels for each factor are widely spread; these are based on regular or non-regular orthogonal fractional factorial 2-level designs. In many applications, there is a need for more than two levels in some factors, even at the screening stage of experimentation. For example, different materials or different geometries might be of interest, and there might be three or four of them that are considered worth to be included into an initial experiment. This article mainly investigates OAs of resolution III or IV, where resolution III implies that main effects and 2-factor interactions can be fully or partially confounded with each other, while resolution IV implies that main effects and 2-factor interactions are orthogonal to each other, but 2-factor interactions may be fully or partially confounded with each other. This concept is well-known for 2-level arrays and is completely analogous for mixed level arrays; for the connection to the strength of an OA, cf. Section 2.

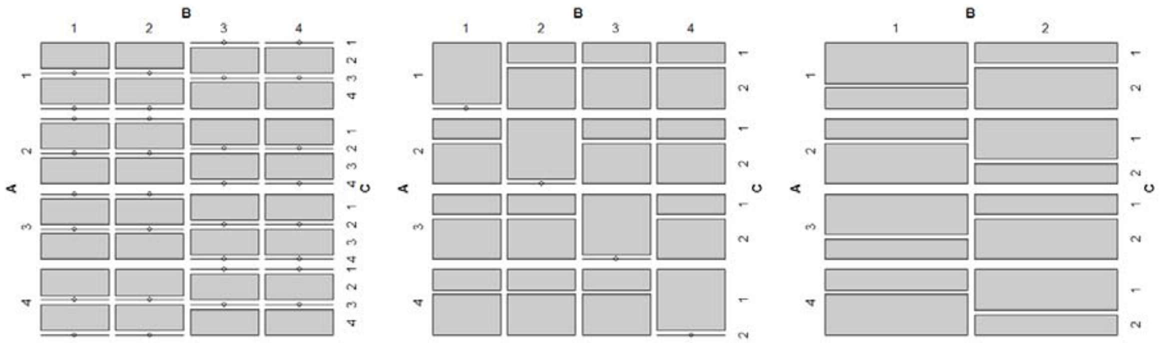
Support for mixed level experiments in statistical software is limited. A few well-researched mixed level OAs like the Taguchi L18 (cf. e.g. NIST/Sematech 2010, section 5.3.3.10) are widely available, for example in Minitab software (Minitab Inc., 2009) or the SAS ADX graphical user interface, which is part of the SAS/QC software (SAS Institute Inc. 2010). A SAS macro suite (Kuhfeld 2009) offers a larger catalogue of general OAs; this suite, like most other implementations of mixed level OAs in software, does not control the statistical properties of the designs it generates, apart from orthogonality of main effects. If mixed level OAs are covered, software usually contains a few arrays with many columns, from which some columns are selected for any particular experiment; this selection is usually not guided by any quality criteria for the design. The research reported here is targeted at improving creation of tailor-made orthogonal arrays in statistical software. The immediate purpose of developing relative projection frequency tables has been to guide selection of columns from a given array for a particular experiment. Automatic creation of arrays in software would also greatly benefit from a more general approach of providing criteria that enable selection of additional OAs for inclusion into the software.

The purpose of relative projection frequency tables as proposed in this article is to provide detailed information about a design's aliasing structure. The extent of complete aliasing of main effects with 2-factor interactions (2fis) in resolution III designs or the extent of complete aliasing of 2fis with each other in resolution IV designs are of particular interest for screening experiments. For illustrating the meaning of "complete aliasing", Figures 1 and 2 present mosaic plots, as introduced by Hartigan and Kleiner (1981). A mosaic plot is very helpful for visualizing the aliasing structure of 3-factor or at most 4-factor projections: The rectangles correspond to the proportions of level combinations; the mosaic plot for a full factorial would look similar to Figure 2 (c), but with equally-sized rectangles. Figure 1 illustrates complete aliasing for fixed and mixed level arrays: Plot (a) shows three completely aliased 4-level factors, for which the level combination of any pair of factors completely determines the level of the third factor. The other two mosaic plots show a design with two 4-level factors and one 2-level factor ((b)) or one 4-level factor and two 2-level factors ((c)). In plot (b), the 2-level factor is completely determined by the level combination of the two 4-level factors, while the combination of any one 4-level factor with the 2-level factor does not completely determine the other 4-level factor. In plot (c), each 2-level factor is determined by the level combination of the 4-level factor and the other 2-level factor, but the 4-level factor is not completely determined by the combination of the 2-level factors. All three graphs depict the most severe aliasing possible within an OA for the respective combination of factor levels. Figure 2 shows mosaic plots for partially aliased OAs for the setups shown in Figure 1. Clearly, these OAs are not perfectly balanced, but they are less aliased than the ones in Figure 1. The degree of partial aliasing differs: plot (a) shows many level combinations that do not occur at all; in plot (b) a few level combinations do not occur at all, but most occur at least once; plot (c) shows all level combinations at least once.



(a) (b) (c)

Figure 1: Mosaic plots for three completely aliased situations



(a) (b) (c)

Figure 2: Mosaic plots for three partially aliased situations

For assessing the severity of aliasing of main effects with 2fis for a particular OA, it is helpful to consider projections of the OA onto any triple of factors – i.e. to investigate the OA after reduction to just the three factors under consideration. Triples with complete aliasing, as depicted in Figure 1, bear a strong risk that an existing 2fi severely biases conclusions on main effects. If possible, this type of aliasing should be avoided in a screening design. The impact of partial aliasing on conclusions for main effects depends on the severity; it is of course desirable to also keep aliasing severity as low as possible.

Projection frequency tables (*PFT*s) provided in the literature (Xu, Cheng and Wu 2004; cf. also next section) generally allow an assessment of the number of projections that are replicates of full factorial designs, even for mixed level arrays. Their proposed modification into relative projection frequency tables (*RPFT*s) will help distinguishing complete aliasing from partial aliasing in mixed level arrays. *RPFT*s also allow a limited assessment of the severity of partial aliasing. *RPFT*s for 3-factor projections can thus be used for selecting OAs with as little bias risk as possible from confounding of main effects with 2fis. It is also possible and relevant to consider *RPFT*s based on 4-factor projections for designs with resolution IV – such designs are also run as screening experiments, if a larger experiment can be afforded but no assumptions on functional form or active interactions are made in advance. This article proposes two scalar metrics in addition to *RPFT*s: The total amount of aliasing will be measured by the proposed scalar metrics rA_3 or rA_4 (or in general rA_R for a resolution R design). The worst-case aliasing and herewith the distance from complete aliasing for the worst-case triple or quadruple of factors will be reflected in the proposed scalar metric *GR*, which is a generalization of generalized resolution by Deng and Tang (1999) to mixed level arrays.

The consideration of orthogonal arrays and their properties may appear obsolete to advocates of D-optimal (or other letter-optimal) designs who might argue that users simply have to specify their model and will receive an optimal design, which will be orthogonal if possible in the specified number of runs. However, especially in screening situations, assuming a model is often not reasonably possible. When assuming a pure main effects model, the D-optimal design will indeed be an orthogonal main effects array, if the number of runs permits. However, D-optimality is not at all influenced by the performance of an OA in terms of behavior of 3-factor projections, i.e. the outcome of a D-optimization is a matter of luck in terms of bias risk. The benefits of using an orthogonal array with reasonable quality criteria, especially for screening, lie in model robustness of the design and further usability of its outcomes if it turns out that only a few of the original factors are of relevance. If orthogonality is desired – which has e.g. been argued for by Kuhfeld and Tobias (2005) – D-optimization is not able to provide the most suitable design in terms of screening properties.

The next section will combine the formalization of the basic concepts underlying this article's results with a review of the literature related to assessing confounding structures of mixed level OAs. Section 3 will motivate *RPFT*s by looking at the established assessments for screening properties in case of 2-level arrays. Section 4.1 will derive the worst-case number of absolute words; technical details of the derivation are deferred to the Appendix. Based on this result, Section 4.2 will develop and exemplify *RPFT*s and will introduce the scalar metric rA_R, which denotes the total number of *relative* words of length R. Section 4.3 will use *RPFT*s for introducing a second scalar metric, the afore-mentioned version of generalized resolution (*GR*) for mixed level OAs. This *GR* also provides a necessary condition for projectivity in the sense of Box and Tyssedal (1996). Section 4.4 will introduce and discuss relative projection aberration. Finally, Section 5 will discuss the limitations of the results, relations to other concepts, needs for further research and implications for statistical software.

2. Setting the scene

Consider an OA in n runs and k factors with the j-th factor at s_j levels, $j = 1,\ldots, k$. W.l.o.g., it is assumed throughout this article that $s_1 \leq s_2 \leq \ldots \leq s_k$. An OA is also denoted as OA(n, $l_1^{k_1}\ldots l_m^{k_m}$), with $k = k_1+\ldots+k_m$ and k_j of the factors at l_j levels, where $l_1 \leq \ldots \leq l_m$ denote the distinct numbers of levels. The degrees of freedom (df) for each factor's main effect are one less than the factor's number of levels, and the total main effects df amount to

$$\sum_{j=1}^{m} k_j(l_j - 1) = \sum_{j=1}^{k} s_j - k\ . \tag{1}$$

All OAs are balanced in the sense that each column contains each number of levels the same number of times, and each pair of columns contains each pair of levels the same number of times. If each group of c columns of the OA contains each level combination the same number of times, the OA is said to be of strength c. In the statistical literature, resolution is a common equivalent expression: an OA of strength c has resolution c+1, where resolution is usually denoted by a roman numeral. For example, a strength 3 OA has resolution IV and contains each triple of levels the same number of times for each triple of factors; this implies the afore-mentioned fact that main effects and 2fis cannot be confounded with each other.

2.1. GWLP and generalized minimum aberration

In a seminal paper, Xu and Wu (2001) introduced the generalized word length pattern (GWLP) and the concept of generalized minimum aberration. These will be introduced in detail shortly. For regular fractional factorial designs, the GWLP coincides with the more well-known word length pattern (WLP, cf. e.g. Mee 2009, Section 5.2), for which all entries are integers. The entries of GWLP can also take non-integer values. For introducing GWLP, it is necessary to look at the model matrix **M** of a full model with all interactions up to the highest possible degree included. In the literature on generalized minimum aberration, it has become customary to divide **M** into portions according to the degree of interactions, i.e.

$$\mathbf{M} = (\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2,\ldots, \mathbf{M}_k), \tag{2}$$

where $\mathbf{M}_0$ is the column of ones for the constant, $\mathbf{M}_1$ holds all the main effects columns (i.e. its number of columns is given by (1)), $\mathbf{M}_2$ contains the columns for the 2fis, and so forth, until $\mathbf{M}_k$ contains the columns for the interaction among all k factors. The columns of $\mathbf{M}_2$ are the pair wise products between columns of $\mathbf{M}_1$ that do not belong to the same factor, the columns for higher order interactions are analogously defined as products of appropriate triples of columns of $\mathbf{M}_1$, and so forth. The specifics of **M** depend on the coding of experimental factors, i.e. on the chosen contrasts, and Xu and Wu (2001) laid out requirements for contrast specification. In this article, these are fulfilled by coding factors in normalized Helmert contrasts, which are proportional to orthonormal Helmert contrasts. The coding of the s–1 columns for an s-level factor ($s \geq 2$) in $\mathbf{M}_1$ is given as

$$\left(\underbrace{-\sqrt{\frac{s}{j(j+1)}} \quad \cdots \quad -\sqrt{\frac{s}{j(j+1)}}}_{j \text{ times}} \quad \sqrt{\frac{js}{j+1}} \quad \underbrace{0 \quad \cdots \quad 0}_{s-j-1 \text{ times}} \right), \; j = 1, \ldots, s-1. \tag{3}$$

The thus-obtained normalized Helmert contrasts for 2, 3 and 4 levels are shown in Table 1. The multiplier $\sqrt{s}$ in each element of (3) ensures compatibility with Xu and Wu's (2001) version of contrast normalization: the Euclidean norm of each main effects column of the model matrix for an n run design with balanced individual columns thus becomes $\sqrt{n}$. This normalization is particularly important for mixed level designs, for which the usual orthonormal Helmert contrasts would yield different Euclidean norms for model matrix columns of factors at different numbers of levels.

Table 1: Normalized Helmert contrasts for factors in 2, 3, and 4 levels

2 levels	3 levels	4 levels
$\begin{array}{r} \text{level 1} \\ \text{level 2} \end{array} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$	$\begin{array}{r} \text{level 1} \\ \text{level 2} \\ \text{level 3} \end{array} \begin{pmatrix} -\sqrt{1.5} & -\sqrt{0.5} \\ +\sqrt{1.5} & -\sqrt{0.5} \\ 0 & \sqrt{2} \end{pmatrix}$	$\begin{array}{r} \text{level 1} \\ \text{level 2} \\ \text{level 3} \\ \text{level 4} \end{array} \begin{pmatrix} -\sqrt{2} & -\sqrt{2/3} & -\sqrt{1/3} \\ +\sqrt{2} & -\sqrt{2/3} & -\sqrt{1/3} \\ 0 & 2\sqrt{2/3} & -\sqrt{1/3} \\ 0 & 0 & \sqrt{3} \end{pmatrix}$

In the following, the model matrix $\mathbf{M}$ will be used for defining the so-called J-characteristics and GWLPs. Subsequently, the GWLPs will be broken down into more detail in a notation closer to statistical modeling conventions. The J-characteristics can be written as the absolute column sums of $\mathbf{M}$, i.e. $\mathbf{J} = |\mathbf{1}_{1\times n}\mathbf{M}|$ is the row vector of all J-characteristics, as introduced by Deng and Tang (1999) for 2-level factors, and named J-characteristics by Tang and Deng (1999). Ai and Zhang (2004) generalized the concept to general OAs. Of course, the J-characteristics depend on the particular coding chosen for the factors. When only looking at the portion of the J-characteristics for a particular degree of interaction, we will e.g. denote $\mathbf{J}_3 = |\mathbf{1}_{1\times n}\mathbf{M}_3|$ for the row vector of J-characteristics relating to 3fis, or abbreviate these as J_3-characteristics.

GWLP = (A_3, A_4, …, A_k) is the vector of numbers of generalized words of lengths 3 to k. The entries of GWLP can be determined from the J-characteristics: $A_f = \mathbf{J}_f\mathbf{J}_f^{\mathrm{T}}/n^2 = \mathbf{1}_{1\times n}\mathbf{M}_f\mathbf{M}_f^{\mathrm{T}}\mathbf{1}_{n\times 1}/n^2$, i.e. e.g. $A_3 = \mathbf{J}_3\mathbf{J}_3^{\mathrm{T}}/n^2 = \mathbf{1}_{1\times n}\mathbf{M}_3\mathbf{M}_3^{\mathrm{T}}\mathbf{1}_{n\times 1}/n^2$. Thus, A_3 can be expressed as the sum of squares of the column sums of $\mathbf{M}_3$, divided by n^2. It is straightforward to verify that this yields the usual WLP for regular fractional factorial 2-level designs. Xu and Wu (2001) showed that the A_f are independent of the factor coding, as long as the effect columns are orthogonal to the intercept column and all main effects model matrix columns are normalized to sum of squares n. Thus, the normalized Helmert contrasts (3) have been chosen for convenience, but the results also hold for any other appropriately normalized contrasts that follow Xu and Wu's instructions. GWLP, like WLP, is directly related to the strength and resolution of an OA: resolution is the smallest word length that occurs with a positive frequency. For example, if A_3>0, resolution is III; for A_3=0 but A_4>0, resolution is IV, and so forth. The resolution of an array will be denoted by R in the following. As mentioned before, the strength of the array is one less than the resolution, i.e. the strength is the largest word length that occurs with zero frequency. Note that resolution is quite different from generalized resolution, which was introduced by Deng and Tang (1999) for non-regular orthogonal arrays with 2-level factors, as

$$GR \quad = R + 1 - \max \left(\mathbf{J}_R \big/ n \right). \tag{4}$$

The purpose of generalized resolution is to indicate, how much the most severely aliased R factor projection of the design deviates from complete aliasing. This concept will also prove useful for mixed level arrays and will be discussed in detail in Sections 3 and 4.3 of this article.

OAs of the same (generalized) resolution can be ranked based on the GWLP according to the (generalized) minimum aberration approach: as proposed by Xu and Wu (2001), the overall best (=generalized minimum aberration) array is the array with highest resolution and fewest shortest (generalized) words. This is completely analogous to the widespread minimum aberration criterion for regular fractional factorial designs. Projection aberration (next section) refines the generalized aberration criterion; the relative projection aberration proposed in this article (cf. Section 4) will reduce the relative weight assigned to interactions among factors with many levels vs. interactions among factors with few levels.

2.2. Projection frequency tables and projection aberration

Projection frequencies as proposed by Xu, Cheng and Wu (2004) split the GWLP into more detailed information: these authors determined separate A_3 values for all projections of a design onto a triple of factors. They called these the "projected A_3 values" and their counts the "projection frequencies". Here, the combination of the projected A_3 values with their counts is called the projection frequency table, abbreviated as *PFT* and more specifically denoted as PFT_3, if restricted to 3-factor projections. Xu, Cheng and Wu proposed to successively minimize the number of projections with worst-case A_3 values and called this criterion the "projection aberration criterion". For example, Table 2 shows PFT_3 for three non-isomorphic 18 run arrays (the third of which is the well-known Taguchi L18), which Schoen (2009) ranked according to this criterion: All three arrays have one 3-factor projection with three 3-level factors and complete aliasing. They differ in the number of second-worst 3-factor projections with one generalized word of length 3 and are therefore ranked as shown in the table.

Table 2: PFT_3 for the three OA(18,$2^1 3^7$) of Schoen (2009)

$a_3(u,v,w)$	0	4/9	½	2/3	1	2	A_3
first	9	9	16	21	0	1	28
second	10	6	20	17	2	1	28
third	12	0	28	9	6	1	28

In the following, frequency tables for generalized words of length f for projections onto f factors, called PFT_fs, are formally derived, in preparation for introducing relative projection frequency tables. The case f=3 is most important; nevertheless, f=4 and larger f are also covered. The main effects model matrix $\mathbf{M}_1$ consists of k individual model matrices $\mathbf{X}_1, \ldots, \mathbf{X}_k$ with s_j–1 columns for the j-th matrix. Analogously, each portion of $\mathbf{M}$ with a higher index than 1 can be subdivided into columns that belong to particular factorial interaction effects. Notationally, we will in the following consider the u-th, v-th and w-th factor, for 4fis in addition the t-th factor, $u<v<w<t$; remember that $s_u \le s_v \le s_w$ ($\le s_t$). Let $\mathbf{X}_u$ and $\mathbf{X}_v$ denote the main effects matrix of the u-th and v-th factor, $\mathbf{X}_{uv}$ the matrix for the 2fi between the u-th and v-th factor, $\mathbf{X}_{uvw}$ the matrix for the 3fi between the u-th, v-th and w-th factor. Then, $\mathbf{M}_2$ consists of the matrices $\mathbf{X}_{uv}$ for all pairs (u,v), $\mathbf{M}_3$ of the matrices $\mathbf{X}_{uvw}$ for all triples (u,v,w), and so forth. Obviously, $A_3 = \mathbf{1}_{1\times n}\mathbf{M}_3\mathbf{M}_3^{\mathrm{T}}\mathbf{1}_{n\times 1}/n^2$ can be split into summands relating to individual triples of factors. If the projected A_3 value for the projection onto the u-th, v-th and w-th factor is denoted as

$$a_3(u,v,w) = \mathbf{1}_{1\times n}\mathbf{X}_{uvw}\mathbf{X}_{uvw}^{\mathrm{T}}\mathbf{1}_{n\times 1}/n^2, \tag{5}$$

we obtain

$$A_3 = \sum_{u=1}^{k-2}\sum_{v=u+1}^{k-1}\sum_{w=v+1}^{k} a_3(u,v,w). \tag{6}$$

PFT_3 is the frequency table of the $\binom{k}{3}$ individual $a_3(u,v,w)$. Analogously, for 4 factors, defining $\mathbf{X}_{uvwt}$ as the model matrix of the 4fi for the u-th, v-th, w-th and t-th factor, $a_4(u,v,w,t) = \mathbf{1}_{1\times n}\mathbf{X}_{uvwt}\mathbf{X}_{uvwt}^{\mathrm{T}}\mathbf{1}_{n\times 1}/n^2$ is the projected A_4 value for the projection onto these four factors, A_4 is the sum over all quadruples (u,v,w,t) of the $a_4(u,v,w,t)$, and PFT_4 is the frequency table of the $\binom{k}{4}$ individual $a_4(u,v,w,t)$. Generally, PFT_f refers to the frequency table of length f generalized words for projections onto f factors. Thus, PFT_f is a more detailed version of A_f. Note that the PFTs are related to Deng and Tang's (1999) confounding frequency vectors for fixed 2-level arrays, and Deng and Tang's "minimum G aberration" criterion is equivalent to the projection aberration criterion. Tang and Deng (1999) proposed a simplified "minimum G2 aberration" criterion, which Xu and Wu (2001) showed to be a special case of generalized minimum aberration.

As was mentioned before, the most important PFT is PFT_3. For fixed level OAs of s level factors, PFT_3 directly reveals how many 3-factor projections are completely aliased and thus bear the risk of biasing main effects by 2fis: as will be shown in Section 4.1, the number of completely aliased 3-factor projections is the number of 3-factor projections with s–1 (generalized) words of length 3. For mixed-level OAs, *PFT*s are not as easily interpretable, as the number of generalized words of length 3 that corresponds to complete aliasing of 3-factor projections depends on the numbers of levels for each of the 3 factors. This is the reason for proposing relative *PFT*s for mixed level arrays. These will be introduced in Section 4.

3. Usefulness of PFTs for screening properties of 2-level arrays

If all factors have 2 levels in a regular array, there are only two possibilities for the u-th, v-th and w-th factor: either the design confounds the main effect of the u-th factor with the 2fi of the v-th and w-th factor, in which case $\mathbf{X}_{uvw}$ is a constant column of "+1" only or "–1" only entries; or there is no aliasing for these three factors, in which case $\mathbf{X}_{uvw}$ is a column with half the entries "+1" and half the entries "–1". Hence, $a_3(u,v,w)$ can be either 1 or 0, and PFT_3 delivers the counts of triples being completely aliased or not aliased at all. For non-regular fixed level arrays with 2-level factors, a single length 3 word can be distributed over several projections onto three factors. Table 3 provides an example: the regular fractional factorial array for 14 factors in 16 runs is compared to the irregular 16 run array for 14 2-level factors, which was proposed in Box and Tyssedal (2001) based on projectivity considerations as well as in Deng and Tang (2002) based on minimum G aberration. Both arrays have A_3=28 (generalized) words of length 3. Table 3 shows that the regular array distributes its A_3 words over 28 fully-aliased 3-factor projections, keeping all other 3-factor projections unaliased, while the non-regular array more evenly distributes the A_3 words: 112 of the 364 3-factor projections are affected by partial aliasing.

Table 3: PFT_3 for the regular and non-regular 16 run array for 14 factors

$a_3(u,v,w)$	0	0.25	1	A_3
Regular array	336	0	28	28
Non-regular array	252	112	0	28

The example of Table 3 illustrates the idea of using PFT_3 as a tool for assessing suitability of an array for screening purposes. The regular array has 28 completely aliased 3-factor projections with the full risk that 2fis bias conclusions on main effects. The non-regular array has partially aliased 3-factor projections only with less severe individual bias risks (but – of course – more of those). Apart from the bias risk, it is a further advantage of the non-regular array that all projections onto triples of factors would be able to separate main effects from 2fis in a subsequent analysis, should a main effects analysis point to the particular triple of factors as being important.

For fixed level arrays with 2-level factors, Deng and Tang (1999) proposed the scalar criterion *GR*, which was already introduced in (4). Their idea was to increase the resolution R by the deviation of worst case aliasing among R factors from 100%. For example, in a resolution III design, if any 3-factor projection is completely aliased, generalized resolution is 3. If, however, the worst case aliasing is partial only, generalized resolution is larger than 3: the baseline 3 is increased by the gap between worst case aliasing in the design and 1, where worst case aliasing is measured in terms of the maximum of the normalized *J*-characteristics. In the notation of this article, (4) can be expressed as

$$GR = R + 1 - \max_{\{c_1,\dots,c_R\}\subset\{1,\dots,k\}}\left(\sqrt{a_R\left(c_1,\dots,c_R\right)}\right), \tag{7}$$

where R is the resolution of the design, $c_1<\ldots<c_R$ are indices of R distinct design columns, and $a_R(c_1,\ldots,c_R)$ is the number of generalized words of length R for the projection onto the R factors indexed by these indices. It is important to note that (7) is only valid for 2-level arrays. Absence of complete aliasing is directly visible from *GR*. For example, the non-regular array of Table 3 has $GR = 3.5$, because its worst-case number of length 3 words is 0.25, which is the square of 0.5; the corresponding regular array has generalized resolution 3 only, because the largest a_3 is 1. For designs with resolution R (= strength R–1), projections onto R–1 factors are (replicated) full factorials. For designs in 2-level factors, Deng and Tang (1999, their proposition 2) showed that a non-integer $GR > R$ implies that a projection onto R factors contains at least $n(GR–R)/(2^R)$ full factorials in the R factors plus the remaining runs as replicates of half-fractions. Thus, for 2-level factors, $GR > R$ implies projectivity R in terms of Box and Tyssedal (1996).

This section used *PFT*s in absolute terms for 2-level designs; as each factorial effect has only one df in this situation, PFT_f contains equivalent information to a table of normalized J_f-characteristics. This is not true if any factors have more than 2 levels, since several J_f-characteristics are related to one factorial effect in that case.

4. RPFTs and relative projection aberration

In the previous section, *PFT*s have been used for assessing the suitability of a 2-level array for screening. Apart from generalized resolution, the concepts carry over easily to fixed level arrays at $s > 2$ levels, as all 3-factor projections are comparable to each other in terms of the numbers of levels of their factors.

Projection aberration as proposed by Xu, Cheng and Wu (2004) sequentially minimizes the frequency of projections with the worst case number of shortest (generalized) words. For mixed level arrays, if interest is in ensuring absence of complete aliasing, it is not generally adequate to compare unadjusted frequencies between projections onto f factors, because the number of length f words associated with complete aliasing depends on the pattern of numbers of levels. For example, one word of length 3 implies complete aliasing for three 2-level factors, but is different from complete aliasing for three 3-level factors. If an array contains various 2-level and 3-level factors, some projections with one word of length 3 may be completely aliased while others are only partially aliased. For $RPFT_R$s, the number of words of length R for each projection onto R factors is normalized such that 1=100% corresponds to complete aliasing. It is proposed to select screening arrays using the “relative projection aberration” criterion based on $RPFT_R$, i.e. to sequentially minimize the frequency of projections with the worst case *relative* numbers of shortest (generalized) words. This procedure should be applied after selecting an appropriate design according to the scalar criteria that will be proposed below. with the overall lowest number of relative words Furthermore, $RPFT$s can also be used for obtaining a generalization of (7).

4.1. The maximum possible number of generalized words

In order to obtain a relative version of PFT, it is crucial to find appropriate normalizing quantities, so that 100% indeed corresponds to complete aliasing for the 3- or 4-factor projections at hand. These quantities will be derived in this section for 3-factor projections in general, for 4-factor projections in strength 3 (=resolution IV) arrays, and generally for R-factor projections in resolution R arrays. The case of projecting onto $f > R$ factors, e.g., 4-factor projections in resolution III arrays, is less useful, because the most interesting worst-case aliasing is already captured by the R-factor projections. It is also more complicated to identify normalizing quantities for this case; this task is therefore left to future research. The result to be derived below is stated here in general terms: For a projection onto R factors with indices $c_1<\ldots<c_R$, sorted such that $s_{c1}\leq\ldots\leq s_{cR}$, the worst case reference for the number of (generalized) words of length R is $w_R(c_1,\ldots,c_R) = s_{c1} - 1$. The result is derived for $R = 3$ and then extended.

Like in Section 2.2, consider a 3-factor projection of an OA with the model matrices $\mathbf{X}_u$, $\mathbf{X}_v$, $\mathbf{X}_w$ for the factors’ main effects, $s_u \leq s_v \leq s_w$ for their numbers of levels, n for the number of runs, and $n_{\text{distinct}(u,v,w)}$ for the number of distinct runs of the 3-factor projection; often $n_{\text{distinct}(u,v,w)} < n$. As the array is orthogonal, $n_{\text{distinct}(u,v,w)}$ must be a multiple of s_us_v, s_us_w and s_vs_w, i.e. it must be at least the least common multiple (LCM) of these products, which can be both larger than or equal to s_vs_w. (Note that no assumptions have been made that any number of levels is a prime.) Complete aliasing among three factors is possible, if LCM(s_us_v, s_us_w, s_vs_w) = s_vs_w = $n_{\text{distinct}(u,v,w)}$. In this case, the main effect of the u-th factor can be completely aliased with the 2fi between the v-th and w-th factor, which happens if and only if the combination of levels of the v-th and w-th factor fully determines the level of the u-th factor (cf. e.g. the plots in Figure 1). It is shown in the Appendix, that the number of generalized words of length 3 in this case is s_u–1, the df for the main effect of the factor with the fewest levels. This result also holds more generally: the worst-case number of generalized words in an R-factor projection for a resolution R array is the df of the factor with the fewest levels among the R factors. The proof in the appendix is detailed for resolution III arrays and sketched for resolution IV arrays. It is straightforward but notationally more complex to generalize it to resolution R.

The worst case number of words has been derived for cases, where $n_{\text{distinct}(c1,\ldots,cR)}$ is the product of the R–1 larger numbers of levels, i.e. $n_{\text{distinct}(c1,\ldots,cR)} = s_{c2}\ldots s_{cR}$. It has been mentioned that the worst case is not possible for all situations. For example, with one 4-level factor, one 3-level factor and one 2-level factor, i.e. $s_u = 2$, $s_v = 3$, $s_w = 4$, the above-derived worst case would be $s_u–1 = 1$. However, an OA requires at least 24 runs, i.e. as many runs as a full factorial. The worst allocation of combinations to 24 runs, which is compatible with orthogonality, leads to 2/3 generalized length 3 words only – better than the theoretical worst case of 1. This practically-attainable worst case OA for $s_u = 2$, $s_v = 3$, $s_w = 4$ has an aliasing behavior (not shown), which is clearly different from complete aliasing. As the purpose for deriving a point of reference is to indicate severity of the consequences for experimentation, it is considered appropriate to use a reference that reflects complete aliasing even though this is not practically attainable under all circumstances.

4.2. RPFTs and rA_R

For $RPFT_3$, each number $a_3(u,v,w)$ of generalized words of length 3 of a particular 3-factor projection is divided by its respective worst-case number $w_3(u,v,w)$, as derived in the previous section. Thus, the relative frequency of generalized words of length 3 in the projection onto the u-th, v-th and w-th factor is given as $r_3(u,v,w) = a_3(u,v,w) / w_3(u,v,w) = a_3(u,v,w) / (s_u–1)$. Generally, for projections onto the R factors indexed by $c_1 < c_2 < \ldots < c_R$, again ordered with increasing numbers of levels,

$$r_R(c_1,\ldots,c_R) = a_R(c_1,\ldots,c_R)/w_R(c_1,\ldots,c_R) = a_R(c_1,\ldots,c_R)/(s_{c_1}-1). \tag{8}$$

If interest is in relative rather than absolute projection frequencies, an overall assessment of the extent of aliasing in the design can be obtained by obtaining the sum over all $r_R(c_1,\ldots,c_R)$, thus obtaining the overall number rA_R:

$$rA_R = \sum_{\substack{c_1<\ldots<c_R \\ \{c_1,\ldots,c_R\}\subseteq\{1,\ldots,k\}}} r_R(c_1,\ldots,c_R) \tag{9}$$

Table 4 gives $RPFT_3$ and rA_3 for the resolution III 18 run arrays of Table 2. From the absolute numbers of generalized words of length 3 in Table 2, it is obvious that the only projection with 2 generalized words of length 3 must be from three 3-level factors; this is the only completely-aliased projection. By comparing Tables 2 and 4, it can be concluded that the 35 3-factor projections with 0.5, 1 and 2 generalized words of length 3 come from projections with 3-level factors only, while the other 21 3-factor projections include the 2-level factor. For all three arrays, the overall absolute A_3 of 28 reduces to the overall relative rA_3 of 17.

Table 4: $RPFT_3$s for the 18 run arrays of Table 2

Relative	0	1/4	1/3	4/9	1/2	2/3	1	rA_3
First	9	16	18	9	0	3	1	17
Second	10	20	12	6	2	5	1	17
Third	12	28	0	0	6	9	1	17

4.3. Generalizing generalized resolution

The example arrays of Table 4 have at least one completely aliased 3-factor projection. Table 5 gives two examples of mixed-level arrays without any complete aliasing: In both cases, the largest relative number of generalized words of length 3 within any 3-factor projection is 2/3. Analogously to generalized resolution for 2-level arrays (cf. (4) and (7)), it would be desirable to reflect this improvement over complete aliasing in a version of generalized resolution for mixed level OAs.

Remember the results on GR shown previously: For resolution III 2-level arrays, the square roots of the $a_3(u,v,w)$ are the normalized J_3-characteristics. They can only take values between 0 and 1, and GR for the 2-level case is defined based on the maximum absolute normalized J_R-characteristic with R denoting the resolution of the design (cf. (4) and (7)). In general, $a_3(u,v,w)$ is the sum of several squared normalized J_3-characteristics and can be larger than one. Thus, it cannot be used directly for an analogous definition of generalized resolution. However, moving from absolute to relative projection frequencies, an ad-hoc generalization could base a generalized resolution for general OAs on the maximum square root of the $r_3(u,v,w)$, which is again guaranteed to be between 0 and 1. While consideration of the square root appears natural in the 2-level case because of Deng and Tang's (1999) geometric result for the addition to the resolution (their proposition 2, cf. Section 3), a convincing motivation for the general case is lacking so far. Nevertheless, analogy implies the following proposal:

$$GR = R + 1 - \max_{\{c_1,\dots,c_R\}\subset\{1,\dots,k\}}\left(\sqrt{r_R\left(c_1,\dots,c_R\right)}\right) \qquad (10)$$

with R the resolution of the design, $c_1<\ldots<c_R$ indices of R distinct design columns, and $r_R(c_1,\ldots,c_R)$ the relative number of (generalized) words of length R of the projection onto the R factors indexed by these indices. For 2-level designs, (10) coincides with (7), as the $r_R()$ and $a_R()$ coincide for all R-factor projections. For mixed-level designs, (10) and (7) also coincide, whenever there are at most $R-1$ factors at more than two levels, because this implies $w_R\left(c_1,\dots,c_R\right)=s_{c_1}-1=1$ for all index sets $\{c_1,\ldots,c_R\}$.

Table 5: Two resolution III mixed level OAs with generalized resolution larger than 3

PFT_3 and $RPFT_3$ for OA(36, $2^{11}3^{12}$), A_3=194, rA_3=184								
$a_3(u,v,w)$	0	1/9	1/8	1/6	1/2	2/3	7/8	
frequency	855	162	192	417	93	33	16	
$r_3(u,v,w)$	0	1/9	1/8	1/6	1/4	7/16	½	2/3
frequency	855	162	192	417	12	16	81	33

PFT_3 and $RPFT_3$ for an OA(18, 2^13^6), A_3=17, rA_3=10.5 (array 6.1.5 of Schoen 2009, isomorphic to Taguchi L18 without 3rd 3-level column)				
$a_3(u,v,w)$	0	1/2	2/3	1
frequency	9	14	6	6
$r_3(u,v,w)$	0	1/4	1/2	2/3
frequency	9	14	6	6

(10) is a reasonable proposal in the following sense: it yields $GR = R$ if and only if the design contains at least one completely aliased projection onto R factors. Thus, $GR > R$ is a necessary condition for projectivity R. Applying GR from (10) to the well-known OA(36, $2^{11}3^{12}$) and a particular OA(18, 2^13^6) (cf. Table 5) yields GR = 3.1835 (= $3+1-\sqrt{2/3}$) in both cases, i.e. these arrays fulfill the necessary condition for projectivity 3. Note, however, that $GR > R$ does not imply projectivity R for general OAs: none of the arrays of Table 5 has projectivity 3.

Another potential way of generalizing Deng and Tang's (1999) generalized resolution would base the assessment on normalized J-characteristics much like in the original definition. The resulting definition for GR would directly apply formula (4). However, this approach does not lead to a reasonable result, as it can and does happen that GR is larger than R even though some 3-factor projections are completely aliased. For example, even the completely aliased array of Figure 1 (a) yields a maximum of absolute normalized J_3-characteristics (calculated using normalized Helmert contrasts according to (3)) of about 0.612, which would imply a generalized resolution of 3.388. This is of course not acceptable, as the level of each factor is completely determined by the level combination of the other two factors, which is appropriately reflected in the fact that the design has the maximum possible number of length 3 words, i.e. s_u–1 = 3. It is not surprising that the formula based on J-characteristics does not easily generalize to mixed level OAs, since J-characteristics are known to depend on the actual parameterization of the experimental factors. The generalization of GR proposed in (10) appears far more appropriate and returns $GR = R = 3$ for all arrays of Figure 1. In comparison, the arrays in Figure 2 have GR = 3.4226 (for (a) and (b)) and GR = 3.6667 (for (c)).

As has been mentioned before, $GR > R$ implies projectivity R for designs with 2-level factors only (Deng and Tang, 1999), but not for general OAs. Deng and Tang's proposition 2 makes (7) appear a natural extension of resolution for the 2-level case. A similarly compelling rationale has not been found for the general case of (10). Choice of the exact formula in (10) therefore appears somewhat arbitrary.

4.4. Relative projection aberration

The relative projection aberration criterion ranks designs according to their *RPFT*. For Table 4, the design order, determined by projection aberration (cf. Section 2.2), remains unchanged for relative projection aberration: All three arrays have one 3-factor projection with complete aliasing. They are ranked based on the numbers of 3-factor projections with 2/3 relative words of length 3 (3 < 5 < 9).

For a reasonable implementation of relative projection aberration, note that first and foremost, complete aliasing is to be avoided, which can be achieved by maximizing *GR*. As $RPFT_R$ is a more detailed version of rA_R, it appears natural to continue by selecting the design with lowest rA_R among the designs with maximum *GR*. Thus, it is proposed to apply relative projection aberration in the following steps:

Step (a): Find the designs with highest possible *GR*.

Step (b): Among these, find the designs with lowest rA_R.

Step (c): Among these, rank according to $RPFT_R$, like in (absolute) projection aberration.

Step (d): Among these, rank with respect to A_{R+1}, A_{R+2}, and so forth.

Steps (a) to (c) are all based on $RPFT_R$. Step (d) is not in line with the logic of the relative approach. As long as ties have to be broken after application of Step (c), this violation of the relative approach cannot be avoided, because there is (currently) no relative metric for projections onto more than *R* factors, as a normalizing quantity has not (yet) been derived. The following two examples demonstrate application of relative projection aberration in comparison to (absolute) projection aberration. In the first example, step (d) leads to a unique choice, in the second it doesn't. The second example demonstrates the beneficial effect of the costly initial *GR* optimization (rather than just watching out for resolution).

Table 6 shows a 32 run array, for which PFT_3 and $RPFT_3$ are shown in Table 7. A computer search for the best (in terms of generalized minimum aberration) allocation of three 2-level factors and five 4-level factors to columns of this array returned eight designs with A_3=20 and A_4=58. These come in two different variants w.r.t. PFT_3 and $RPFT_3$, as shown in the first two designs in Table 8. Clearly, (absolute) projection aberration would prefer design 1, because it has no 3-factor projection with three generalized length 3 words. Applying only step (c) of relative projection aberration to the first two designs of Table 8 would yield a preference for design 2, because it has 9 instead of 10 completely aliased 3-factor projections. The complete relative projection aberration approach yields an even better design in relative terms: Step (a): There is no column allocation with *GR*>3, i.e. all 2520 possible choices of columns are equivalent in terms of *GR*. Step (b): There are six resolution III column allocations with the minimum rA_3 of 35/3. Step (c): All six solutions from

(b) have the same $RPFT_3$. Step (d): Design 3 from Table 8 minimizes A_4 among the six solutions (the other five have A_4=55).

Table 6: OA(32, $2^{10}4^7$) (transposed, columns are the runs, rows the factors)

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2	1	1	1	1	2	2	2	2
1	1	1	1	2	2	2	2	2	2	2	2	1	1	1	1	2	2	2	2	1	1	1	1	1	1	1	1	2	2	2	2
1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2
1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1
1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	2	2	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1
1	2	1	2	1	2	1	2	2	1	2	1	2	1	2	1	1	2	1	2	1	2	1	2	2	1	2	1	2	1	2	1
1	2	2	1	1	2	2	1	2	1	1	2	2	1	1	2	2	1	1	2	2	1	1	2	1	2	2	1	1	2	2	1
1	2	3	4	2	1	4	3	4	3	2	1	3	4	1	2	3	4	1	2	4	3	2	1	2	1	4	3	1	2	3	4
1	2	3	4	2	1	4	3	3	4	1	2	4	3	2	1	1	2	3	4	2	1	4	3	3	4	1	2	4	3	2	1
1	2	3	4	4	3	2	1	2	1	4	3	3	4	1	2	4	3	2	1	1	2	3	4	3	4	1	2	2	1	4	3
1	2	3	4	4	3	2	1	1	2	3	4	4	3	2	1	2	1	4	3	3	4	1	2	2	1	4	3	3	4	1	2
1	2	3	4	3	4	1	2	4	3	2	1	2	1	4	3	4	3	2	1	2	1	4	3	1	2	3	4	3	4	1	2
1	2	3	4	3	4	1	2	3	4	1	2	1	2	3	4	2	1	4	3	4	3	2	1	4	3	2	1	2	1	4	3
1	1	1	1	3	3	3	3	2	2	2	2	4	4	4	4	4	4	4	4	4	2	2	2	2	3	3	3	3	1	1	1	1

The array was created from Kuhfeld's (2009) parent OA(32, $4^8 8^1$) by expanding the 8-level factor into OA(8, $2^4 4^1$) and the first two 4-level factors into OA(4, 2^3) each.

Table 7: PFT_3 and $RPFT_3$ for the OA(32, $2^{10}4^7$) of Table 6 (A_3=148, rA_3=122)

PFT_3	0	1	3	$RPFT_3$	0	1/3	1
	536	142	2		536	33	111

Table 8: PFT_3 and $RPFT_3$ for three OA(32, $2^3 4^5$) created from the OA(32, $2^{10}4^7$) of Table 6

no.	*selected columns*	*GWLP*		PFT_3			$RPFT_3$			rA_3
		A_3	A_4	0	1	3	0	1/3	1	
1	3, 5, 8, 11, 12, 15, 16, 17	20	58	36	20	0	36	10	10	13.33
2	3, 9, 10, 12,13, 14, 15, 17	20	58	38	17	1	38	9	9	12.00
3	1, 3, 4, 13, 14, 15, 16, 17	21	53	38	15	2	38	8	9	11.67

Table 9: PFT_3 and $RPFT_3$ for three OA(18, $2^1 3^6$) created from the Taguchi L18

no.	*omitted 3-level column*	A_3	A_4	PFT_3					rA_3	$RPFT_3$					*GR*
				0	1/2	2/3	1	2		0	1/4	1/2	2/3	1	
1	1st	16	28.5	6	20	9	0	0	11.0	6	20	0	9	0	3.183
2	3rd	17	24.5	9	14	6	6	0	10.5	9	14	6	6	0	3.183
3	7th	17	24.5	9	16	6	3	1	10.5	9	16	3	6	1	3

For the second example, an 18 run design with one 2-level factor and six 3-level factors is to be obtained from the Taguchi L18. There are only seven ways for obtaining such a design, by in turn omitting each of the seven 3-level columns from the L18. Among the 12 existing non-isomorphic OA(18, $2^1 3^6$) (cf. Schoen 2009, Table IV), only the three presented in Table 9 can be obtained from the Taguchi L18. These are best, fifth and last in Schoen's ordering of designs according to projection aberration. Design 1 of Table 9 is the generalized minimum aberration design and has been obtained by optimizing GWLP among the seven designs; design 2 of Table 9 is one of the two isomorphic choices that are obtained by relative projection aberration. Design 3 of Table 9 is an instance of the remaining four designs, all isomorphic to the worst design from Schoen; it was obtained in an initial naïve try of minimizing rA_3 and subsequently minimizing A_4, which was at first considered as a potentially valid and simpler approach than full relative projection aberration. After ending up with design 3, this approach was dropped. Comparing the best design in absolute terms (design 1) to the best design in relative terms (design 2; also coincides with the 18 run design from Table 5), design 2 is better than design 1 in two ways: it confounds only 26 instead of 29 triples of factors, and the most severe partial confounding occurs less frequently (for 6 instead of 9 triples). This change is brought about by a shift of confounding from triples involving the 2-level factor to triples of only 3-level factors; the latter contribute a larger number to A_3 than to rA_3, which accounts for the difference.

5. Discussion

*RPFT*s and *rA* have been developed for projections onto R factors, where R is the resolution. All projections onto $f < R$ factors have of course 0 (generalized) words of length f. For projections onto $f > R$ factors, e.g. 4-factor projections for resolution III designs, the maximum conceivable number of words of the respective length has not been derived so that a reasonable standardization of PFT_f to $RPFT_f$ is not possible, and consequently rA_f can neither be calculated. This limitation is not too severe, since (*R*)*PFT*s are by far most interesting for projections onto R factors. Nevertheless, it might be interesting to provide an analogous scaling for projections onto more than R factors; for example, this might enable the introduction of a relative word length pattern, which could then be used to make step (d) of the relative projection aberration criterion more consistent with the concept of relative consideration (cf. end of Section 4.4). Note that the whole concept of projection aberration and relative projection aberration, like also (generalized) minimum aberration, relies on equal importance of all effects. If certain factors or interactions are of minor importance, these can be intentionally confounded more than others, including complete aliasing.

This article emphasized screening experiments, and thus mostly concentrated on $(R)PFT_3$. Schoen (2010) compared the behavior of D-optimal designs and resolution IV OAs for 29 scenarios for which estimation of 2fis was requested. Six of these gave the same design with both approaches, two arrays were not feasible as OAs (full factorial needed). Among the 21 remaining cases, Schoen found a clear preference for a resolution IV OA (7 cases) or for a D-optimal design (4 cases) or a dependence of the preference on the purpose of the experiment (10 cases). Most of Schoen's example cases have no more than three factors at more than 2 levels so that $(R)PFT_4$ and PFT_4 coincide. Nevertheless, in principle Schoen's investigation lends support to the usefulness of $RPFT_4$ in addition to $RPFT_3$.

The relation of *RPFT*s to two other concepts has been presented in the previous section: they provide a possibility for generalizing generalized resolution and consequently also a necessary condition for projectivity. It would also be interesting to investigate the relation to uniformity of an array (cf. e.g. Fang, Ma and Mukerjee 2002). As an ad-hoc idea, one might e.g. investigate the variability of the $r_3(u,v,w)$ for all triples $u<v<w$, or their concentration, and consider a design the more appropriate for screening, the lower the variability or the concentration. The relation of this proposal to uniformity of a design might be a topic of interest.

Wu and Zhang (1993), before the seminal work by Xu and Wu (2001), considered regular fractional factorial designs with 2-level and 4-level factors, that can be generated from fractional factorial 2-level designs by assigning each 4-level factor to a triple of 2-level factors that share a word of length 3. Realizing that the same number of words of a given length has different implications for aliasing, depending on how many 4-level factors are involved, they proposed to have separate counts for different types of words instead of just one version each of A_3, A_4, … While their idea has some appeal, it adds a lot of complexity and is not easily generalizable to general mixed level OAs. Note that *RPFT*s do not solve Wu and Zhang's (1993) issue: their proposal to differentiate between different types of words in the (generalized) word length pattern for designs with 2-level and 4-level factors has been investigated for one and two 4-level factors only, in which case *RPFT* always coincides with *PFT*.

As was mentioned before, the final goal of this research – from a statistician's perspective – is to achieve increased usability of mixed level OAs with a better understanding of the consequences of aliasing. This entails better inclusion of general orthogonal arrays into statistical software: as mentioned in the introduction, coverage of OAs in commercial software products is limited and could use improvement. For a start, relative projection aberration can be applied for column allocation when picking columns from OAs available in software, as illustrated in Section 4.4. Furthermore, it would be desirable to have software offer a larger range of orthogonal arrays, possibly together with catalogued quality information like resolution, generalized resolution, GWLP, PFT_R, and/or rA_R, $RPFT_R$, and with an algorithm that automatically generates a good – perhaps even the best – orthogonal array according to a reasonable set of quality criteria. There is still a long way to go until such an approach can be put into practice: current efforts into enumerating non-isomorphic OAs yield so many different arrays that even today's computing power does not allow to implement these into routine software. Improvements can perhaps be expected from offline application of quality criteria to large catalogues of non-isomorphic OAs; the most promising such arrays can then be included into software. Choice of adequate criteria and algorithms for tailoring an OA to an experimental situation will eventually ensure choice of a good or even optimal orthogonal array. The concepts proposed here are most likely a start rather than the final solution. The R-package DoE.base (Grömping 2011) implements all these; it is hoped that this implementation stipulates readers to work with the methods which will lead to improvements based on practical experiences with the consequences of the various criteria.

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Appendix

Situation: The projection onto the u-th, v-th and w-th factor at $s_u \le s_v \le s_w$ levels has

$n_{\text{distinct}(u,v,w)} = s_v s_w = \text{LCM}(s_u s_v,\, s_u s_w,\, s_v s_w)$ distinct runs.

The projection is completely aliased, i.e. the level combination of the v-th and w-th factor completely determines the level of the u-th factor.

To be shown: $a_3(u,v,w) = s_u - 1$.

Proof: Orthogonality of the array and the chosen normalized Helmert contrasts imply $\mathbf{X}_v^{\mathrm{T}}\mathbf{X}_v = n\,\mathbf{I}_{sv-1}$, $\mathbf{X}_w^{\mathrm{T}}\mathbf{X}_w = n\,\mathbf{I}_{sw-1}$, $\mathbf{X}_v^{\mathrm{T}}\mathbf{X}_w = \mathbf{0}_{(sv-1)\times(sw-1)}$, and $\mathbf{X}_{vw}^{\mathrm{T}}\mathbf{X}_{vw} = n\,\mathbf{I}_{(sv-1)(sw-1)}$. The last identity holds, because the projection onto the v-th and w-th factor is a (potentially replicated) full factorial.

As the levels of the u-th factor are fully determined by the 2fi between the v-th and w-th factor, $\mathbf{X}_u$ can be written as

$$\mathbf{X}_u = \mathbf{X}_{vw}\,\mathbf{K} \qquad (11)$$

for a suitable s_u–1 column matrix $\mathbf{K}$. Due to usage of normalized Helmert contrasts, $\mathbf{X}_u^{\mathrm{T}}\mathbf{X}_u = n\mathbf{I}_{su-1}$, which together with $\mathbf{X}_{vw}^{\mathrm{T}}\mathbf{X}_{vw} = n\,\mathbf{I}_{(sv-1)(sw-1)}$ implies that $\mathbf{K}^{\mathrm{T}}\mathbf{K} = \mathbf{I}_{su-1}$.

According to (5), $n^2\,a_3(u,v,w) = \mathbf{1}_{1\times n}\mathbf{X}_{uvw}\mathbf{X}_{uvw}^{\mathrm{T}}\mathbf{1}_{n\times 1}$. Now, realize that the columns of $\mathbf{X}_{uvw}$ are element wise products of the columns in $\mathbf{X}_u$, $\mathbf{X}_v$, $\mathbf{X}_w$, which implies

$$\mathbf{1}_{1\times n}\,\mathbf{X}_{uvw} = \left(\sum_{i=1}^{n} \mathbf{X}_{u(i,f)}\mathbf{X}_{\mathrm{v}(i,g)}\mathbf{X}_{\mathrm{w}(i,h)}\right)_{f,g,h}, \qquad (12)$$

where indices (i, j) stand for the i-th row and j-th column, respectively. (Note that the elements of (12) are the unnormalized signed J-characteristics corresponding to the u,v,w interaction.) Exploiting the structure of (12) and the structure of $\mathbf{X}_{vw}$ and inserting (11), formula (12) can be rewritten as

$$\mathbf{1}_{1\times n}\mathbf{X}_{uvw} = \mathrm{vec}(\mathbf{X}_u^{\mathrm{T}}\mathbf{X}_{vw})^{\mathrm{T}} = \mathrm{vec}(\mathbf{K}^{\mathrm{T}}\mathbf{X}_{vw}^{\mathrm{T}}\mathbf{X}_{vw})^{\mathrm{T}} = n\,\mathrm{vec}(\mathbf{K}^{\mathrm{T}})^{\mathrm{T}},$$

where the vec operator stacks the columns of a matrix on top of each other, i.e. generates a column vector from all elements of a matrix. This implies

$$n^2\,a_3(u,v,w) = \mathbf{1}_{1\times n}\mathbf{X}_{uvw}\,\mathbf{X}_{uvw}^{\mathrm{T}}\mathbf{1}_{n\times 1} = n^2\,\mathrm{vec}(\mathbf{K}^{\mathrm{T}})^{\mathrm{T}}\mathrm{vec}(\mathbf{K}^{\mathrm{T}}).$$

Exploiting the relation $\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^{\mathrm{T}}\otimes\mathbf{A})\mathrm{vec}(\mathbf{B})$ for dimensionally suitable matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ (cf. e.g. Bernstein 2009), choosing $\mathbf{A}=\mathbf{B}=\mathbf{I}_{su-1}$ and $\mathbf{C}=\mathbf{K}^{\mathrm{T}}$ yields

$$\begin{aligned}
n^2\,a_3(u,v,w) &= n^2\,\mathrm{vec}(\mathbf{I}_{su-1})^{\mathrm{T}}(\mathbf{K}^{\mathrm{T}}\otimes\mathbf{I}_{su-1})\,(\mathbf{K}\otimes\mathbf{I}_{su-1})\,\mathrm{vec}(\mathbf{I}_{su-1})\\
&= n^2\,\mathrm{vec}(\mathbf{I}_{su-1})^{\mathrm{T}}(\mathbf{K}^{\mathrm{T}}\,\mathbf{K}\otimes\mathbf{I}_{su-1})\,\mathrm{vec}(\mathbf{I}_{su-1})\\
&= n^2\,\mathrm{vec}(\mathbf{I}_{su-1})^{\mathrm{T}}\mathrm{vec}(\mathbf{I}_{su-1})\\
&= n^2(s_u-1).
\end{aligned}$$

This proves the assertion. ///

For generalization to the worst case number of (generalized) words in 4-factor projections for resolution IV arrays, note that strength 3 = resolution IV implies that the projection onto any three factors is a (potentially replicated) full factorial. This implies

- that the matrix $\mathbf{X}_{vwt}$ for the 3-factor interaction of the v-th, w-th and t-th factor fulfills $\mathbf{X}_{vwt}^{\mathrm{T}}\mathbf{X}_{vwt} = n\,\mathbf{I}_{(sv-1)(sw-1)(st-1)}$ (again relying on the Helmert contrast coding according to (3)),
- and that $n_{\mathrm{distinct}(u,v,w,t)}$ must be a multiple of $s_us_vs_w$, $s_us_vs_t$, $s_us_ws_t$, $s_vs_ws_t$, so that $\mathrm{LCM}(s_us_vs_w, s_us_vs_t, s_us_ws_t, s_vs_ws_t) \le n_{\mathrm{distinct}(u,v,w,t)} \le s_us_vs_ws_t$.

  This makes $n_{\mathrm{distinct}(u,v,w,t)} = s_vs_ws_t$ the worst-case number of runs.

Complete aliasing implies that the main effect of the u-th factor (s_u–1 df) is completely determined by the 3fi between the v-th, w-th and t-th factor. With exactly the same reasoning as for the 3-factor projections, $\mathbf{X}_u$ can be written as $\mathbf{X}_{vwt}\,\mathbf{K}$ for a suitable matrix $\mathbf{K}$ which implies that the number of length 4 words in this setup is $a_4(u,v,w,t) = s_u-1$, i.e. we have again the smallest number of levels reduced by 1. The general resolution R case is completely analogous but notationally more complex.