Übungsblatt 9
Gewöhnliche Differentialgleichungen

Gewöhnliche Differentialgleichungen, Grundbegriffe:

A1.

Prüfen Sie zu der gegebenen DGL, ob die gegebene Funktion die Lösung der DGL ist:

$$x = \ln y' + \sin y', \qquad x = \ln t + \sin t, \quad y = t(1 + \sin t) + \cos t$$

Gewöhnliche Differentialgleichungen 1. Ordnung

Integration durch Trennung der Variablen

A2

1. $2(x-1)\,dx + 3y^2\,dy = 0, \qquad y(2) = 1$
2. $y' = x(y-1), \qquad y(0) = \dfrac{1}{2}$
3. $y' + x^2(y-2) = 0, \qquad y(0) = -2$

A3

1. $y' - x^2 = 0, \qquad a)\; y(-3) = 1, \quad b)\; y(2) = 5$
2. $y' = x^2 + 4x - 3, \qquad a)\; y(1) = -2, \quad b)\; y(-3) = 3$
3. $y' = 4 + y, \qquad a)\; y(0) = 1, \quad b)\; y(1) = 3$

Integration durch Substitution

A4:

1. $y' = 4x - y, \qquad a)\ y(2) = 4, \qquad b)\ y(0) = -2$
2. $y' = x + y, \qquad a)\ y(0) = 2, \qquad b)\ y(-2) = 3$

Variation der Konstanten

A5

1. $y' + y = m\,e^{-nx}, \qquad y(0) = 1,$
   $a)\ m = n = 1, \qquad b)\ m = 1,\ n = 2, \qquad c)\ m = 2,\ n = 1$
2. $y' + \frac{y}{x} = \sin(mx), \qquad y(\pi) = 1, \qquad a)\ m = 1, \qquad b)\ m = 2$
3. $xy' + my = x, \qquad y(1) = -1, \qquad a)\ m = 1, \qquad b)\ m = 2, \qquad c)\ m = 4$
4. $xy' + my = x^2, \qquad y(1) = 0, \qquad a)\ m = 2, \qquad b)\ m = -2$
5. $xy' - y = x^m, \qquad y(1) = 1, \qquad a)\ m = 1, \qquad b)\ m = 2, \qquad c)\ m = 3$

Lösungen – Übungsblatt 9 :

1.

Gewöhnliche Differentialgleichungen, Grundbegriffe:

$x = \ln y' + \sin y', \qquad x = \ln t + \sin t, \quad y = t(1 + \sin t) + \cos t$

$$y' = y'_x = \frac{y'(t)}{x'(t)}$$

$$x'(t) = \frac{1}{t} + \cos t, \qquad y'(t) = 1 + \sin t - \sin t + t(1 + \cos t) = 1 + t\cos t$$

$$y' = \frac{1 + t\cos t}{\frac{1}{t} + \cos t} = \frac{1 + t\cos t}{\frac{1}{t}(1 + t\cos t)} = t, \quad t > 0$$

$$\Rightarrow \quad \ln t + \sin t = \ln t + \sin t$$

Lsg.

2.

Gewöhnliche Differentialgleichungen 1. Ordnung

Integration durch Trennung der Variablen

1. $2(x-1)\,dx + 3y^2\,dy = 0, \qquad y(2) = 1$

$$2\int (x-1)\,dx + 3\int y^2\,dy = 0, \qquad (x-1)^2 + y^3 = C$$

Allgemeine Lösung: $(x-1)^2 + y^3 = C$

$y(2) = 1: \quad C = 2, \qquad (x-1)^2 + y^3 = 2$

2. $y' = x(y-1), \qquad y(0) = \frac{1}{2}$

$$\int \frac{dy}{y-1} = \int x\,dx, \quad \ln|y-1| = \frac{x^2}{2} + \ln|C|, \quad \ln\left|\frac{y-1}{C}\right| = \frac{x^2}{2}$$

Allgemeine Lösung: $y = e^{\frac{x^2}{2}} + C$

$y(0) = \frac{1}{2}: \quad C = -\frac{1}{2}, \quad y = e^{\frac{x^2}{2}} - \frac{1}{2}$

Noch zu Lsg.2:

3. $y' + x^2 (y-2) = 0, \qquad y(0) = -2$

$$\int \frac{dy}{y-2} = -\int x^2\,dx, \qquad \ln|y-2| = -\frac{x^3}{3} + \ln|C|$$

Allgemeine Lösung: $y = 2 + C\,e^{-\frac{x^3}{3}}$

$y(0) = -2\ :\quad C = -4, \qquad y = 2 - 4\,e^{-\frac{x^3}{3}}$

Lsg.

3.

1. $y' - x^2 = 0, \qquad a)\ y(-3) = 1, \quad b)\ y(2) = 5$

Allgemeine Lösung: $y = \dfrac{x^3}{3} + C$

$a)\ y(-3) = 1\ :\quad y = \dfrac{x^3}{3} + 10, \qquad b)\ y(2) = 5\ :\quad y = \dfrac{x^3}{3} + \dfrac{7}{3}$

2. $y' = x^2 + 4x - 3, \qquad a)\ y(1) = -2, \quad b)\ y(-3) = 3$

Allgemeine Lösung: $y = \dfrac{x^3}{3} + 2\,x^2 - 3\,x + C$

$a)\ y(1) = -2\ :\quad y = \dfrac{x^3}{3} + 2\,x^2 - 3\,x - \dfrac{4}{3}$

$b)\ y(-3) = 3\ :\quad y = \dfrac{x^3}{3} + 2\,x^2 - 3\,x - 15$

3. $y' = 4 + y, \qquad a)\ y(0) = 1, \quad b)\ y(1) = 3$

Allgemeine Lösung: $y = C\,e^x - 4$

$a)\ y(0) = 1\ :\quad y = 5\,e^x - 4, \qquad b)\ y(1) = 3\ :\quad y = \dfrac{7}{e}\,e^x - 4 = 7\,e^{x-1} - 4$

Lsg.

4. Integration durch Substitution

1. $y' = 4x - y, \qquad a)\ y(2) = 4, \qquad b)\ y(0) = -2$

$$u = 4x - y, \qquad \frac{du}{dx} = 4 - \frac{dy}{dx} \quad \Leftrightarrow \quad u' = 4 - y'$$

$y' = 4x - y = u$ in Eq. $u' = 4 - y'$ einsetzen: $u' = 4 - u$

$$\frac{du}{dx} = 4 - u, \qquad \int \frac{du}{4-u} = \int dx, \qquad \ln(4-u) = x + \ln C, \qquad 4 - u = C\,e^x$$

weiter folgt die Rücksubstitution

Allgemeine Lösung: $y = C\,e^{-x} + 4x - 4$

$a)\ y(2) = 4: \quad y = 4x - 4, \qquad b)\ y(0) = -2: \quad y = 2\,e^{-x} + 4x - 4$

2. $y' = x + y, \qquad a)\ y(0) = 2, \qquad b)\ y(-2) = 3$

Allgemeine Lösung: $y = C\,e^{x} - x - 1$

$a)\ y(0) = 2: \quad y = 3\,e^{x} - x - 1, \qquad b)\ y(-2) = 3: \quad y = 2\,e^{x+2} - x - 1$

Lsg.

5. Variation der Konstanten

1. $a)\ m = n = 1: \quad y' + y = e^{-x}, \qquad y = (x + C)\,e^{-x}$

$y(0) = 1: \quad y = (1 + x)\,e^{-x}$

$b)\ m = 1,\ n = 2: \quad y' + y = e^{-2x}, \qquad y = (-e^{-x} + C)\,e^{-x}$

$y(0) = 1: \quad y = (2 - e^{-x})\,e^{-x}$

$c)\ m = 2,\ n = 1: \quad y' + y = 2\,e^{-x}, \qquad y = (2x + C)\,e^{-x}$

$y(0) = 1: \quad y = (1 + 2x)\,e^{-x}$

2. a) $m = 1: \quad y' + \frac{y}{x} = \sin x, \qquad y = \frac{1}{x}(\sin x - x\cos x + C)$

$y(\pi) = 1: \quad y = \frac{1}{x}(\sin x - x\cos x)$

b) $m = 2: \quad y' + \frac{y}{x} = \sin(2x), \qquad y = \frac{1}{4x}(\sin(2x) - 2x\cos(2x) + 4C)$

$y(\pi) = 1: \quad y = \frac{1}{4x}\sin(2x) - \frac{1}{2}\cos(2x) + \frac{3\pi}{2x}$

3. a) $m = 1: \quad xy' + y = x, \qquad y = \frac{x}{2} - \frac{C}{x}, \qquad y(1) = -1: \quad y = \frac{x}{2} - \frac{3}{2x}$

b) $m = 2: \quad xy' + 2y = x, \qquad y = \frac{x}{3} + \frac{C}{x^2}, \qquad y(1) = -1: \quad y = \frac{x}{3} - \frac{4}{3x^2}$

c) $m = 4: \quad xy' + 4y = x, \qquad y = \frac{x}{5} + \frac{C}{x^4}, \qquad y(1) = -1: \quad y = \frac{x}{5} - \frac{6}{5x^4}$

4. a) $m = 2: \quad xy' + 2y = x^2, \qquad y = \frac{x^2}{4} + \frac{C}{x^2}, \qquad y(1) = 0: \quad y = \frac{x^2}{4} - \frac{1}{4x^2}$

b) $m = -2: \quad xy' - 2y = x^2, \qquad y = (\ln x + C)x^2, \qquad y(1) = 0: \quad y = x^2 \ln x$

5. a) $m = 1: \quad xy' - y = x, \qquad y = (\ln x + C)x, \qquad y(1) = 1: \quad y = (\ln x + 1)x$

b) $m = 2: \quad xy' - y = x^2, \qquad y = (x + C)x, \qquad y(1) = 1: \quad y = x^2$

c) $m = 3: \quad xy' - y = x^3, \qquad y = \frac{x^3}{2} + Cx, \qquad y(1) = 1: \quad y = \frac{x^3}{2} + \frac{x}{2}$